Geometric group theory
Lecture 14

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1 Bass–Serre theory

Amalgamated free products and HNN extensions can be viewed as the most basic examples of a much larger and unified theory of combination constructions, which is intimately linked to the theory of group actions on trees. This theory is commonly called *Bass–Serre theory*; it arose initially in Serre's study of the group $\mathrm{PSL}_2(\mathbb{Q}_p)$ – a p-adic linear group whose Bruhat–Tits building is a tree, and was expanded on by Bass.

Since its inception, Bass–Serre theory has become an increasingly essential tool in geometric group theory and low-dimensional topology. A frequent use is to decompose given infinite groups into simpler pieces glued together in certain ways, which are often easier to understand individually.

We will need a little bit of terminology to be able to proceed. Recall that for us, a graph Γ is a set of *vertices* $V\Gamma$ and *edges* $E\Gamma$, together with initial and terminal vertex maps $\iota, \tau \colon E\Gamma \to V\Gamma$, and en edge inversion map $\bar{\cdot} \colon E\Gamma \to E\Gamma$, which has the property that $\bar{\bar{e}} = e, \iota(\bar{e}) = \tau(e)$ and $\tau(\bar{e}) = \iota(e)$.

Definition 1.1 (Graph of groups)**.** A *graph of groups* is a tuple $\mathcal{G} = (\Gamma, G_-, \varphi_-)$ where Γ is a connected graph, G_v and G_e are groups for each $v \in V\Gamma$ and $e \in E\Gamma$, and $\varphi_e \colon G_e \to G_v$ is an injective homomorphism for each $e \in E\Gamma$. We require that for each $e \in E\Gamma$, we have an equality of groups $G_e = G_{\bar{e}}$. We call the groups G_v the *vertex groups*, G_e the *edge groups*, and φ_e the *edge morphisms*.

Similarly to how one defines the fundamental group of a space as the set of loops on a given basepoint up to homotopy, we can define a fundamental group of a graph of groups as the set of loops in the underlying graph, with the extra data that paths can pick up group elements from vertex spaces and that 'homotopy' will respect edge morphisms. One should think of this construction as gluing together the vertex groups along the edge groups in a way prescribed by the edge morphisms.

Definition 1.2 (Fundamental group of a graph of groups)**.** Let $\mathcal{G} = (\Gamma, G_-, \varphi_-)$ be a graph of groups, and pick a basepoint $v_0 \in \Gamma$. Let $F(\mathcal{G})$ be the quotient of the free product $F(E\Gamma) * (\ast_{v \in V\Gamma} G_v)$, subject to the relations $e\bar{e} = \bar{e}e = 1$ and $e\varphi_e(g)\bar{e} = \varphi_{\bar{e}}(g)$ for all $e \in E\Gamma$ and $g \in G_e$.

The *fundamental group* $\pi_1(\mathcal{G}, v_0)$ of $\mathcal{G}$ based at v_0 is the subgroup of $F(\mathcal{G})$ consisting of (images of) words of the form $g_0 e_1 g_1 \dots e_n g_n$ where $e_1 \dots e_n$ forms a loop based at v_0, and $g_i \in G_{v_i}$ for each $i = 0, \dots, n$, with $v_i = \tau(e_i)$ when $i > 0$.

Of course, the isomorphism type of the fundamental group of a graph of groups is independent of the choice of basepoint in the underlying graph. Indeed, similarly to fundamental groups of spaces, changing the basepoint amounts to conjugating in the auxiliary group $F(\mathcal{G})$. We will thus often suppress mention of the basepoint and simply write $\pi_1\mathcal{G}$. We can now realise the constructions of the previous lecture as exactly the fundamental groups of one-edge graphs of groups.

Example 1.3 (Amalgamated free product)**.** Let Γ be the graph consisting of a single edge e with distinct endpoints u and v. Let G_u, G_v, and G_e be groups with injective homomorphisms $\varphi_e \colon G_e \to G_u$ and $\varphi_{\bar{e}} \colon G_e \to G_v$, and consider the graph of groups $\mathcal{G} = (\Gamma, G_-, \varphi_-)$. Picking u as a basepoint, the only loops in Γ are powers of $e\bar{e}$. Then

$$F(\mathcal{G}) \cong \langle G_u, G_v, e \mid e\varphi_e(g)e^{-1} = \varphi_{\bar{e}}(g), g \in G_e \rangle,$$

so that $\pi_1\mathcal{G}$ is exactly the subgroup generated by G_u and $eG_v e^{-1}$, which is easily seen to be isomorphic to $G_u *_{G_e} G_v$.

Example 1.4 (HNN extension)**.** Let Γ be the graph consisting of a single edge e with $v = \iota(e) = \tau(e)$. That is, Γ is a single edge loop on a vertex v. Let G_v be a group, $G_e \leqslant G_v$ a subgroup and $\varphi_e \colon G_e \to G_v$ an injective homomorphism. Let $\mathcal{G} = (\Gamma, G_-, \varphi_-)$ be the corresponding graph of groups, considering $\varphi_{\bar{e}} \colon G_e \to G_v$ as the inclusion map of G_e as a subgroup. The loops in Γ are exactly the powers of e, so in this case $\pi_1\mathcal{G} = F(\mathcal{G})$. Hence we have the presentation $\pi_1\mathcal{G} = \langle G_v, e \mid ege^{-1} = \varphi_e(g), g \in G_e \rangle$, from which we can exactly see that $\pi_1\mathcal{G} \cong G_v *_{\varphi_e}$.

Remark 1.5. The fundamental group of any graph of groups can be realised as an iterated sequence of amalgamated free products and HNN extensions, by collapsing a single edge or a loop at a time. Thus statements about fundamental groups of graphs of groups can be proven by induction, with these single-edge cases as the base cases.

Fundamental groups of graphs of groups also admit normal forms, similarly to amalgamated products and HNN extensions. As in those settings, an immediate consequence is that the vertex groups canonically embed into the fundamental group.

Theorem 1.6. *Let $\mathcal{G} = (\Gamma, G_-, \varphi_-)$ be a graph of groups, and suppose that $e_1 \dots e_n$ is a loop in Γ based at $v_0 \in V\Gamma$. If $g_0 e_1 g_1 \dots e_n g_n = 1$ in $\pi_1(\mathcal{G}, v_0)$, where $g_i \in G_{v_i}$ and $v_i = \tau(e_i)$, then either*

1. *$n = 0$ and $g_0 = 1$; or*
2. *there is $i = 1, \dots, n-1$ such that $e_i = \overline{e_{i+1}}$ and $g_i \in \varphi_{e_i}(G_{e_i})$.*

Exercise 1.7. Show that if $\mathcal{G}$ is a graph of groups, any finite subgroup of $\pi_1\mathcal{G}$ is conjugate into a vertex or edge group of $\mathcal{G}$.

The universal cover of a graph, in the traditional sense, is a tree, and the action of the fundamental group of the graph on this tree via deck transformations recovers the original graph. Analogously, one can build a sort of equivariant universal covering tree for a graph of groups, which admits an action of its fundamental group that will recover the original graph of groups as a quotient. The action of a fundamental group of a graph via deck transformations on its universal cover is free — it has no fixed points. By contrast, the action of a fundamental group of a graph of groups on its universal covering tree will in general have many point stabilisers, which will exactly recover the vertex and edge groups of the original graph of groups.

Definition 1.8 (Bass–Serre tree)**.** Let $\mathcal{G} = (\Gamma, G_-, \varphi_-)$ be a graph of groups. We construct a graph $T_\mathcal{G}$ with a $\pi_1\mathcal{G}$-action, called the *Bass–Serre tree* of $\mathcal{G}$.

The vertices of $T_\mathcal{G}$ will be the cosets of vertex groups of $\mathcal{G}$ in $\pi_1\mathcal{G}$, and the edges of $T_\mathcal{G}$ will likewise be cosets of edge groups of $\mathcal{G}$ in $\pi_1\mathcal{G}$. We must define the incidence relations and inversion map. Given an edge $gG_e \in ET_\mathcal{G}$, where $e \in E\Gamma$, we define $\overline{gG_e} = gG_{\bar{e}}$, $\iota(gG_e) = gG_{\iota(e)}$, and $\tau(gG_e) = gG_{\tau(e)}$. The $\pi_1\mathcal{G}$ action on $T_\mathcal{G}$ is given by the permutation action on cosets that comprise the vertices and edges of $T_\mathcal{G}$.

It is straightforward to check that this action preserves the graph structure, and that the stabilisers of vertices (respectively, edges) are exactly the conjugates in $\pi_1\mathcal{G}$ of the vertex (respectively, edge) groups of the graph of groups $\mathcal{G}$. By definition, there is one orbit of vertices for each vertex group of $\mathcal{G}$ and one orbit of edges for each edge group of $\mathcal{G}$. Moreover, an edge of $T_\mathcal{G}$ is incident to a vertex of $T_\mathcal{G}$ if and only if the corresponding cosets of the edge group and vertex group are defined on an edge incident to a vertex in Γ. It follows that, as a graph, $T_\mathcal{G}/\pi_1\mathcal{G}$ is isomorphic to Γ.

We will not prove it here, but the Bass–Serre tree of a graph of groups is indeed a tree in the sense that it is a connected graph with no non-trivial cycles. This construction has an obvious converse.

Definition 1.9 (Quotient graph of groups)**.** Let G be a group and T a tree. Suppose that G acts on T without edge inversions — that is, there is no $e \in ET$ and $g \in G$ such that $ge = \bar{e}$ – so that $\Gamma = T/G$ is a well-defined graph.

Let $s\colon \Gamma \to T$ be a section of the quotient map $T \to \Gamma$. For each $v \in V\Gamma$, let G_v be the stabiliser in G of $s(v) \in VT$, and likewise for each $e \in E\Gamma$, define G_e to be the stabiliser of $s(e) \in E\Gamma$. Moreover, the edge morphisms φ_e are defined as the inclusion maps $G_e \hookrightarrow G_v$ of edge stabilisers into their adjacent vertex stabilisers (after possibility conjugating, if one of the endpoints of an edge is outside the image of the section s). The resulting graph of groups $\mathcal{G} = (\Gamma, G_-, \varphi_-)$ is called the *quotient graph of groups* of T.

Note that acting without edge inversions is not a serious restriction, since we can always subdivide an edge if there are edge inversions. The fundamental structure theorem of Bass–Serre theory tells us that the these above constructions cohere; if G acts on a tree T, then T is in fact the Bass–Serre tree of for the quotient graph of groups of T.

Theorem 1.10. *Let G be a group acting on a tree T without edge inversions, and let $\mathcal{G}$ be the quotient graph of groups. Then G is isomorphic to $\pi_1\mathcal{G}$ and there is a G-equivariant isomorphism between T and the Bass–Serre tree $T_\mathcal{G}$ of $\mathcal{G}$.*

Example 1.11. The modular group $G = \mathrm{SL}_2(\mathbb{Z})$ admits an action on the hyperbolic plane $\mathbb{H}^2$ by Möbius transformations. Its fundamental domain is a triangle with one vertex at infinity, and the group acts by translations and inversions permuting a tiling of $\mathbb{H}^2$ by copies of this triangle. Take the arc C between the midpoint of the finite side of this triangle and one of the adjacent vertices. The graph formed by the translates $G \cdot C$ is a graph T with two orbits of vertices and one orbit of edges. As T consists of the union of all finite edges of the tiling and each triangle has only one such edge, T is in fact a tree. The vertex stabilisers are $\mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$, and the edge stabiliser is $\mathbb{Z}/2\mathbb{Z}$. It follows that G is isomorphic to the amalgamated free product $\mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$.

Exercise 1.12. Show that the fundamental group of a graph of finite groups has a finite index free subgroup. In particular, $\mathrm{SL}(2, \mathbb{Z})$ has a finite index free subgroup.

The correspondence given by Bass–Serre theory can also be used to give some sleek proofs of appealing statements.

Theorem 1.13. *Let G be a group with a free action on a tree. Then G is a free group.*

Proof. Let G act on a tree T freely, and write $\Gamma = T/G$. The action is without edge inversion, as otherwise it would fix a midpoint of an edge. Then by the structure theorem $G \cong \pi_1\mathcal{G}$, where $\mathcal{G}$ is the quotient graph of groups. Since the action of G was free, the vertex and edge groups of $\mathcal{G}$ are trivial. It follows that $\pi_1\mathcal{G} \cong \pi_1\Gamma$ is a free group. □

The following states that a subgroup of a free product is again a free product, whose factors are a free group and conjugates of subgroups of the original free factors. It is rather a pain to prove this without Bass–Serre theory.

Theorem 1.14 (Kurosh subgroup theorem)**.** *Let G and H be groups, and let K be a subgroup of the free product $G * H$. Then there are collections of subgroups $\{G_i \mid i \in I\}$ and $\{H_j \mid j \in J\}$, conjugate into G and H respectively, and a subset X of $G * H$ such that*

$$K = \Big(\ast_{i\in I}\, G_i \Big) * \Big(\ast_{j\in J}\, H_j \Big) * F(X).$$

Proof. Let T be the Bass–Serre tree of the free product $G * H$, viewed as the fundamental group of the single-edge graph of groups with trivial edge group and vertex groups G and H. Now $G * H$ acts on T without edge inversion, so K does as well. Let T_K be a minimal K-invariant subtree of T under this action. Then we see that $K = \pi_1\mathcal{K}$, where $\mathcal{K}$ is the quotient graph of groups of T_K. The edge groups are trivial, since the edge stabilisers in T were trivial. It follows that K is a free product of the vertex groups of $\mathcal{K}$ together with $\pi_1\Gamma$, where $\Gamma = T_K/K$. The vertex groups of $\mathcal{K}$ are all contained in vertex stabilisers of the action of $G * H$ on T, so they are conjugate into G or H as required. □

2 Accessibility and further decompositions

The dictionary between graphs of groups and actions on trees provided by Bass–Serre theory gives us a powerful language to speak about groups with. For instance, the rather clunky statement of Stallings' theorem on ends of groups may be restated thus:

Theorem 2.1. *If G is a finitely generated group with more than one end, then G acts non-trivially on a tree with finite edge stabilisers.*

If the vertex stabilisers of the action given by the above theorem are again many-ended, then one can apply the theorem once more to obtain a more refined decomposition of the original group. In theory, this process may never terminate; to say it does means that there is a finite graph of groups with finite edge groups and zero- or one-ended vertex groups.

Definition 2.2 (Accessibility)**.** Let G be a finitely generated group. We say that G is *accessible* if it acts on a tree T with finite edge stabilisers, vertex stabilisers with at most one end, and T/G a finite graph.

One would hope that this is true for every finitely generated group, but there are many examples, the first of which was constructed by Dunwoody, of finitely generated groups that are not accessible. Nevertheless, most countable groups one might care about are accessible.

Theorem 2.3 (Dunwoody, '85)**.** *Let G be a finitely presented group. Then G is accessible.*

In particular, we saw that hyperbolic groups are finitely presented, hence accessible. Note that the statement of Dunwoody is actually a little more general: being finitely presented is in some sense a homotopical condition, and the statement holds for groups satisfying a (strictly weaker) homological analogue of finite presentability.

To prove the above, one constructs finite cut sets on the universal cover of a Cayley complex that are in some sense minimal. Dual to this collection of cut sets is a tree that is acted on by the group in the appropriate way.

Knowing that a group is accessible reduces many questions one could ask about that group to its finitely many one-ended 'factors'. For hyperbolic groups, the vertex groups in a graph of groups with finite edge groups are in fact quasiconvex, and so are also hyperbolic. Thus many things one could ask about the class of hyperbolic groups reduces to questions about one-ended hyperbolic groups (that is, hyperbolic groups with connected boundaries). For these, the next least complicated possible decompositions after splittings over finite groups are also understood: those where the edge groups are infinite virtually cyclic. Again, one can detect the splitting directly from connectedness properties of the boundary; recall that a *local cut point* of a topological space is one that disconnects at least one of its neighbourhoods.

Theorem 2.4 (Bowditch, '97)**.** *A one-ended hyperbolic group G acts on a tree with infinite virtually cyclic edge stabilisers if and only if ∂G contains a local cut point.*

The proof involves constructing a tree out of the local cut point structure of ∂G, which admits a canonical G-action. This resulting splitting is usually known as the *JSJ decomposition* of G, in rough analogy with an identically named decomposition in the theory of 3-manifolds.