Geometric group theory
Lecture 4

Lawk Mineh

October 2025

1 More on hyperbolic metric spaces

Proof of Morse lemma (second half): Previously, we showed that there was $M' = M'(\lambda, c, \delta)$ such that $p \subseteq N_{M'}(\gamma(I))$. To show the converse inclusion, let $t \in J$. Of course, if t is an endpoint of J we are done, so suppose otherwise. By continuity of γ', there is a point z of p and numbers $s < t < s'$ such that $\mathrm{d}(\gamma'(s), z) \leq M'$ and $\mathrm{d}(\gamma'(s'), z) \leq M'$. Thus $\mathrm{d}(\gamma'(s), \gamma'(s')) \leq 2M'$. As γ' is a quasi-geodesic, this implies

$$\ell(\gamma'|_{[s,s']}) \leq 2\lambda M' + \lambda c'.$$

Whence $\mathrm{d}(\gamma(t), p) \leq (2\lambda + 1)M' + \lambda c'$. Finally, setting $M = (2\lambda + 1)M' + (\lambda + 1)c'$ completes the proof. $\square$

Note that the key point in the proof was the exponential length estimate on non-geodesic paths in hyperbolic metric spaces. The condition of quasigeodesicity gives a linear upper bound to this exponential bound, but the proof works just as well if the distortion on the length of the path is subexponential.

The most important consequence of the Morse Lemma is the quasi-isometry invariance of hyperbolicity.

Theorem 1.1. *For any $\lambda \geq 1, c \geq 0$, and $\delta \geq 0$ there is a constant $\delta' \geq 0$ such that the following is true.*

Let X and Y be geodesics space, and suppose that X is a δ-hyperbolic metric space. If $f \colon Y \to X$ is a (λ, c)-quasi-isometric embedding, then Y is a δ'-hyperbolic metric space.

Proof. Let Δ be a geodesic triangle in Y with sides p, q, and r. The paths $f(p), f(q)$, and $f(r)$ are (λ, c)-quasi-geodesics in X. Since X is δ-hyperbolic, the Morse Lemma gives us a constant $M = M(\lambda, c, \delta) \geq 0$ such that each of these paths is within an M-Hausdorff neighbourhood of any geodesic between their endpoints.

As geodesic triangles are δ-slim in X, any side of a geodesic triangle whose vertices are the endpoints of $f(p), f(q)$, and $f(r)$ is contained in a δ-neighbourhood of the other two. Thus $f(p)$ is contained in a $(\delta + 2M)$-neighbourhood of $f(q) \cup f(r)$. Let x be a point in p. By the above there is a point y in q or r such that $\mathrm{d}_X(f(x), f(y)) \leq \delta + 2M$. Now as f is a (λ, c)-quasi-isometry, we have

$$\mathrm{d}_Y(x, y) \leq \lambda\, \mathrm{d}_X(f(x), f(y)) + \lambda c \leq (\delta + 2M + c)\lambda.$$

By symmetry, analogous inequalities are true for points on q and on r. Hence every geodesic triangle in Y is $(\delta + 2M + c)\lambda$-slim. We see that Y is δ'-hyperbolic, where $\delta' = 2(\delta + 2M + c)\lambda$. □

Corollary 1.2. *Let X and Y be quasi-isometric geodesic spaces. Then X is a hyperbolic metric space if and only if Y is a hyperbolic metric space.*

Note that δ-hyperbolicity is not invariant under quasi-isometry – the constant of hyperbolicity may change depending on the quasi-isometry constants. Indeed, given a δ-hyperbolic metric space X, one can attach a sphere of diameter R to obtain a space X' that is not δ-hyperbolic if $R > \delta$. Of course, X and X' are $(1, \pi R)$-quasi-isometric, and X' is clearly δ'-hyperbolic, where $\delta' = \delta + R$.

One of the central features of hyperbolicity is the prevalence of many 'local-to-global' phenomena: results that conclude something about the large-scale, global geometry of a space from small-scale, local conditions. Frequently useful is the sufficient condition for quasi-geodesics below.

Definition 1.3 (Local quasi-geodesic)**.** Let X be a metric space, $\lambda \geq 1, c \geq 0$, and $k \geq 0$. A rectifiable path $p\colon I \to X$ is a *k-local (λ, c)-quasi-geodesic* if each subpath q of p with $\ell(q) \leq k$ is a (λ, c)-quasi-geodesic.

Theorem 1.4. *Let X be a δ-hyperbolic space, $\lambda \geq 1$, and $c \geq 0$. There are $\lambda' \geq 1, c' \geq 0$, and $k \geq 0$ such that every k-local (λ, c)-quasigeodesic is (λ', c')-quasigeodesic.*

Proof. Let $M = M(\lambda, c, \delta)$ be the constant of the Morse Lemma, and let $k = 2\lambda(2M + 4\delta + c + 1)$. Let $\gamma\colon I \to X$ be a k-local (λ, c)-quasigeodesic, and let $p\colon J \to X$ be a geodesic with the same endpoints. Suppose both γ and p are parametrised by arc-length. Increasing c, we may assume γ is a continuous path, so that the upper bound $\mathrm{d}(\gamma(t), \gamma(t')) \leq |t - t'|$ holds trivially for any $t, t' \in I$. The idea is that points at $k/2$-intervals along γ will project to points that make uniform progress along on p.

We first claim that $\gamma(I)$ is contained in a uniform neighbourhood of $p(J)$. Let $t \in I$ maximise $\mathrm{d}(\gamma(t), p(J))$, and let $s, u \in I$ be the points $s = \max\{t - \frac{k}{2}, \inf(I)\}$ and $u = \min\{t + \frac{k}{2}, \sup(I)\}$. Let $s', u' \in J$ be such that $p(s')$ and $p(u')$ are the closest points on p to $\gamma(s)$ and $\gamma(u)$. By the Morse Lemma, $\gamma(t)$ is M-close to a point x on a geodesic $[\gamma(s), \gamma(u)]$. Consider the rectangle whose vertices are $\gamma(s), \gamma(u), p(s')$, and $p(u')$. By hyperbolicity, this rectangle is 2δ-slim, so x is 2δ-close to a point w on $[\gamma(s), p(s')], [\gamma(u), p(u')]$, or $[p(s'), p(u')]$. We will rule out the former two possibilities, which completes the claim.

Indeed, suppose that w is a point on $[\gamma(s), p(s')]$ with $\mathrm{d}(x, w) \leq 2\delta$. Thus we have $\mathrm{d}(\gamma(t), w) \leq 2\delta + M$. Since γ is a (λ, c)-quasigeodesic when restricted to $[s, t]$, we have

$\mathrm{d}(\gamma(s), \gamma(t)) \geq \frac{1}{2\lambda}k - c$. The triangle inequality then gives us

$$-\,\mathrm{d}(\gamma(s), w) \leq 2\delta + M - \frac{1}{2\lambda}k + c. \tag{1.1}$$

Now using the fact that $[\gamma(s), p(s')]$ is a geodesic on which w lies,

$$\begin{aligned}\mathrm{d}(\gamma(t), p(s')) &\leq \mathrm{d}(\gamma(t), w) + \mathrm{d}(w, p(s')) \\ &\leq 2\delta + M + \mathrm{d}(\gamma(s), p(s')) - \mathrm{d}(\gamma(s), w).\end{aligned}$$

Whereby (1.1) and the choice of k allows us to conclude

$$\mathrm{d}(\gamma(t), p(s')) \leq \mathrm{d}(\gamma(s), p(s')) + 4\delta + 2M + c - \frac{1}{2\lambda}k < \mathrm{d}(\gamma(s), p(s')),$$

which contradicts the choice of t. A symmetrical argument applies to show that w does not lie on $[\gamma(u), p(u')]$. Therefore $\mathrm{d}(\gamma(t), p(J)) \leq M + 2\delta$.

We are now ready to prove the main statement. Let $t_0 < \cdots < t_n$ be a partition of I such that $t_i - t_{i-1} = k/2$ for $i = 1, \ldots, n-1$, and $t_n - t_{n-1} \leq k/2$. For each $i = 0, \ldots, n$, let $s_i \in J$ be a point with $\mathrm{d}(\gamma(t_i), p(s_i)) \leq 2\delta + M$, as guaranteed to exist by the claim above. By techniques similar to the proof of the claim, one can show $s_{i-1} < s_i$ for all $i = 1, \ldots, n$. Moreover, by local quasigeodesicity of γ and the choice of k,

$$\mathrm{d}(p(s_{i-1}), p(s)) \geq \frac{1}{2\lambda}k - c - 2M - 4\delta \geq 1,$$

for any $i = 1, \ldots, n$. As $s_0 < \cdots < s_n$, it follows that for any $0 \leq i < j \leq n$

$$\mathrm{d}(p(s_i), p(s_j)) \geq j - i.$$

Let $t, t' \in I$ be arbitrary. There are $i, j \in \mathbb{N}$ minimising $|t - t_i|$ and $|t' - t_j|$. By construction, these quantities are at most $\frac{1}{4}k$, and $|t_i - t_j| = \frac{1}{2}|i - j|k$ (out of laziness, we ignore the edge case that i or j is equal to n here). Combining all of the above

$$\begin{aligned}\mathrm{d}(\gamma(t), \gamma(t')) &\geq \mathrm{d}(\gamma(t_i), \gamma(t_j)) - \mathrm{d}(\gamma(t), \gamma(t_i)) - \mathrm{d}(\gamma(t'), \gamma(t_j)) \\ &\geq \mathrm{d}(p(s_i), p(s_j)) - 2M - 4\delta - \frac{1}{2}\lambda k - 2c \\ &\geq |i - j| - 2M - 4\delta - \frac{1}{2}\lambda k - 2c \\ &\geq \frac{2}{k}|t_i - t_j| - c' \\ &\geq \frac{1}{\lambda'}|t - t'| - c',\end{aligned}$$

where $\lambda' = \frac{1}{2}k$ and $c' = 2M + 4\delta + 2c + \frac{1}{2}\lambda k$. □

There are many refinements one can apply to the above statement. For instance, the multiplicative constant λ' can be made arbitrarily close to λ, provided one insists that the local quasigeodesics are quasigeodesic on a sufficiently large scale.

Remark 1.5. Interestingly, the converse of the above theorem also holds: if a geodesic space has the property that all (large enough scale) local quasigeodesics are in fact quasigeodesics, the space is hyperbolic. Thus such local-to-global properties are not only characteristic of hyperbolicity, but unique to it. Indeed, this is true for a number of the basic theorems one proves about hyperbolicity. For instance, a geodesic space where the Morse Lemma holds is also necessarily a hyperbolic metric space.

We state without proof a useful and philosophically important result about hyperbolic spaces. It says that any finite set of points in a hyperbolic space can be approximated uniformly well by a tree having those points as vertices. This adds to the intuition that hyperbolic spaces are really like thickened trees.

Theorem 1.6. *Let X be a δ-hyperbolic metric space. There is a function $h\colon \mathbb{N} \to [0, \infty)$ such that if $x_1, \dots, x_n \in X$ are points, there is an embedded simplicial tree $T \subseteq X$ with $x_1, \dots, x_n$ as vertices with*

$$\mathrm{d}_T(x_i, x_j) \leq \mathrm{d}_X(x_i, x_j) + \delta h(n)$$

for any $i, j = 1, \dots, n$. Moreover, $h(n) = O(\log n)$.

2 Quasiconvexity

In the study of geodesic metric spaces generally, the most well-behaved subspaces are the *convex subspaces*, subspaces that contain every geodesic with endpoints in the subspace. For a convex subspace, the intrinsic length metric of the subspace naturally coincides with the induced metric it inherits from the ambient space, so that the geometry of the subspace respects the geometry of the space it lives in. Since we are interested in coarse geometric properties of metric spaces, we will require a coarse version of this notion.

Definition 2.1 (Quasiconvex subset)**.** Let X be a geodesic space and $\sigma \geq 0$. We say that a subspace $Y \subseteq X$ is *σ-quasiconvex* if the image of any geodesic in X with endpoints in Y is contained in a σ-neighbourhood of Y.

It is a general fact that quasiconvex subsets, equipped with an intrinsic metric, are quasi-isometrically embedded in their ambient spaces. This mirrors the fact that convex subspaces are isometrically embedded in geodesic spaces.

Lemma 2.2. *Let X be a geodesic space with metric d_X, and $Y \subseteq X$ a geodesic subspace with intrinsic length metric d_Y. If there is $\sigma \geq 0$ such that Y is σ-quasiconvex, then the inclusion map $(Y, \mathrm{d}_Y) \to (X, \mathrm{d}_X)$ is a $(2\sigma + 1, 1)$-quasi-isometry.*

Proof. Let $a, b \in Y$ be points, $d = \mathrm{d}_Y(a, b)$. Let $p\colon I \to X$ be a geodesic in X with endpoints a and b. Take a partition $t_0 < \cdots < t_n$ of I where $t_i - t_{i-1} = 1$ for $i = 1, \dots, n-1$ and $t_n - t_{n-1} \leq 1$. We have that $n - 1 \leq \mathrm{d}_X(a, b) \leq n$. For each $i = 0, \dots, n$, there is $y_i \in Y$ with $\mathrm{d}_X(y_i, p(t_i)) \leq \sigma$. Of course, we can take $y_0 = a$ and $y_n = b$.

Now for each $i = 1, \dots, n$, there is a path q_i in Y with endpoints y_{i-1} and y_i such that $\ell(q_i) = \mathrm{d}_Y(y_{i-1}, y_i)$. By construction, $\ell(q_i) \leq 2\sigma + 1$ for each $i = 2, \dots, n-1$ and $\ell(q_1), \ell(q_n) \leq \sigma + 1$. The concatenation q of $q_1, \dots, q_n$ is thus a path in Y with endpoints a and b and $\ell(q) \leq (2\sigma + 1)(n-1) + 1$. It follows:

$$\mathrm{d}_Y(a, b) \leq \ell(q) \leq (2\sigma + 1)\,\mathrm{d}_X(a, b) + 1.$$

The inequality $\mathrm{d}_X(a, b) \leq \mathrm{d}_Y(a, b)$ is immediate, as every path in Y is a path in X. Hence the inclusion map $Y \to X$ is a $(2\sigma + 1, 1)$-quasi-isometric embedding. $\square$

Lemma 2.3. *Let X be a hyperbolic metric space, $Y \subseteq X$ a geodesic subspace. If Y is quasiconvex, then it is a hyperbolic metric space.*

Proof. This is an immediate consequence of Lemma 2.2 and Theorem 1.1. $\square$

In hyperbolic spaces, the converse of Lemma 2.2 also holds true.

Lemma 2.4. *Let X be a δ-hyperbolic metric space, $\lambda \geq 1$ and $c \geq 0$. If Y is a geodesic space and $f\colon Y \to X$ is a quasi-isometric embedding, then there is $\sigma = \sigma(\lambda, c, \delta) \geq 0$ such that $f(Y)$ is σ-quasiconvex.*

Proof. Let $M = M(\lambda, c, \delta)$ be the constant of the Morse Lemma. Any two points in Y may be joined by a geodesic, so that any two points in $f(Y)$ may be joined by a (λ, c)-quasigeodesic lying entirely in Y. The Morse Lemma implies that any geodesic in X with the same endpoints as such a quasi-geodesic is contained in an M-neighbourhood of it. Hence $f(Y)$ is σ-quasiconvex with $\sigma = M$. $\square$