Geometric group theory
Lecture 6

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1 Negative curvature in groups

In this section we will begin to apply the machinery we have developed for metric negative curvature to group theory. Hyperbolic groups are the groups that act geometrically on proper hyperbolic metric spaces. Remarkably, this entirely geometric condition has incredibly strong consequences for the algebraic structure of these groups.

The class of hyperbolic groups includes many groups of classical interest to group theorists, topologists, and geometers. As well as providing a convenient and clarifying framework for understanding groups with 'negative curvature', the study of hyperbolic groups has paved the way for some deep insights into and novel results on these groups.

Definition 1.1 (Hyperbolic group)**.** A group is called *hyperbolic* if it admits a geometric action on a proper hyperbolic metric space.

Equivalently, we may say a finitely generated group is hyperbolic if it has a finite generating set with respect to which the Cayley graph is a a hyperbolic metric space by invoking the Milnor–Schwarz lemma. We saw in the first lecture that quasi-isometry type of a Cayley graph is preserved by a change of finite generating set, and in the fourth lecture that hyperbolicity is preserved by quasi-isometries. Hence the hyperbolicity of any such Cayley graph is independent of which finite generating set is chosen for a hyperbolic group.

Example 1.2.

- The Cayley graph of any finitely generated free group with respect to a free generating set is a simplicial tree, and hence 0-hyperbolic. Therefore finitely generated free groups are hyperbolic.
- If M is a closed hyperbolic n-manifold, its fundamental group $\pi_1 M$ acts geometrically on its isometric universal cover $\mathbb{H}^n$. We saw earlier that $\mathbb{H}^n$ is a hyperbolic metric space, so $\pi_1 M$ is a hyperbolic group.
- Every finite group is hyperbolic as it is quasi-isometric to a point, and every virtually cyclic group is hyperbolic as it is quasi-isometric to a line. We call these *elementary* hyperbolic groups – they are the only virtually abelian ones – and all

others *non-elementary*. We will later see that all non-elementary hyperbolic groups contain non-abelian free subgroups, so are very far from being virtually abelian.

- So-called 'random groups' are hyperbolic. More precisely, one can formulate models of randomness that allow one to choose a finite presentation 'uniformly randomly' in some sense. In most of these models, the 'generic' group is almost always a hyperbolic group.
- A group given by a presentation with relators that do not overlap too much is hyperbolic. Such 'small cancellation' groups are a rich source of examples in geometric group theory, and can exhibit somewhat peculiar properties. We will discuss this class of groups a little later in the course.

Example 1.3. The Cayley graph of $\mathbb{Z}^n$ with respect to the standard generators is not a hyperbolic metric space for any $n \geq 2$, and so $\mathbb{Z}^n$ is not a hyperbolic group.

We will see there is a sort of strong converse to the above example, in that hyperbolic groups cannot contain higher rank abelian groups. This, among other things, will be a consequence of the following important fact.

Theorem 1.4. *Every infinite order element of a hyperbolic group is loxodromic.*

Proof. Let S be a finite generating set for the group G such that $\Gamma(G, S)$ is δ-hyperbolic, and let g be an element of infinite order. Let N be the number of group elements h with $|h|_S \leq 2\delta$, of which there are finitely many. We will show that for any $R \in \mathbb{N}$, we have $\left|g^{RN}\right|_S \geq R$. It follows immediately that g^N is a loxodromic, as it implies

$$|n - m| \leq \mathrm{d}_S(g^{nN}, g^{mN}) \leq |g|_S \cdot |n - m|.$$

That g is a root of a loxodromic then implies g is a loxodromic, so it remains only to prove the claim.

Let $R \in \mathbb{N}$ and take $k \in \mathbb{N}$ large enough so that $\left|g^k\right|_S \geq 4R+2\delta+1$. Now if $|g^n|_S \leq R$, then the geodesic $[1, g^k]$ and its g^n-translate $[g^n, g^{n+k}]$ are at length at least $4R + 2\delta$ and have endpoints at most R apart. We leave it as a straightforward exercise in hyperbolic geometry that the midpoint of the latter path is a distance of at most 2δ from a point on the former path that is at most $\frac{1}{2}R$ away from the midpoint of the former path. There are at most RN such points, by the definition of N. As g does not fix any points in $\Gamma(G, S)$, the midpoints of $[g^n, g^{n+k}]$ must all be distinct. By the pigeonhole principle, then, there is some $n(R) \leq RN+1$ with $\left|g^{n(R)}\right|_S > R$. It follows also that $R \leq n(R)|g|_S$.

Suppose now that $\left|g^{RN}\right| \leq R - \varepsilon$ for some $\varepsilon > 0$. Let $T = \max\{\left|g^i\right|_S \mid 0 \leq i < RN\}$ and $N' = \lceil \frac{1}{\varepsilon} RNT \rceil$. Then for any $n \geq N'$, we have

$$|g^n|_S \leq \left|g^{RN}\right|_S^p + |g^q|_S \leq pR - p\varepsilon + T \leq pR,$$

where $p, q \in \mathbb{Z}$ are such that $n = pRN + q$ and $0 \leq q < RN$. Let $Q = N'|g|_S$, so that $n(Q) \geq N'$. It follows that $\left|g^{n(Q)}\right|_S \leq \frac{n(Q)}{RN} R \leq Q$, while the construction of $n(Q)$ gives that $\left|g^{n(Q)}\right|_S > Q$. This is a contradiction, so we must have $\left|g^{RN}\right| \geq R$ $\square$

The geometric condition of hyperbolicity has some strong implications for the algebraic structure of the group. The beginning of this study sees that centralisers of infinite order elements are always virtually cyclic.

Theorem 1.5. *Let G be a hyperbolic group. If $g \in G$ is an element of infinite order, then $[C_G(g) : \langle g \rangle] < \infty$.*

Proof. Take S be a finite generating set for G, so that $\Gamma(G, S)$ is δ-hyperbolic. Let $\lambda \geq 1$ and $c \geq 0$ be constants for which $n \to g^n$ is a (λ, c)-quasi-isometry $\mathbb{Z} \to \Gamma(G, S)$. We consider this map a quasi-geodesic by precomposing it with a quasi-isometry $\mathbb{R} \to \mathbb{Z}$. Let $M = M(\lambda, c, \delta)$ be the constant obtained by the Morse Lemma.

Let $h \in C_G(g)$ be an arbitrary element of the centraliser of g and write $D = |h|_S$. Since g has loxodromic, there is $N \in \mathbb{N}$ such that $\mathrm{d}_S(1, g^n) > 2\delta + 2M + D$ for all $n \geq N$. Let $n \geq N$ and choose geodesics $p_1 = [1, g^{2n}], p_2 = [h, hg^{2n}], q_1 = [1, h]$, and $q_2 = [g^{2n}, hg^{2n}]$. We may take p_2 to be a h-translate of p_1. These four geodesics form a geodesic rectangle in $\Gamma(G, S)$, which is 2δ-slim as $\Gamma(G, S)$ is δ-hyperbolic.

By Theorem 1.4, the points $\{1, g, \ldots, g^{2n}\}$ are the image of a (λ, c)-quasi-geodesic. Therefore by the Morse Lemma, they lie in an M-Hausdorff neighbourhood of p_1. As $p_2 = hp_1$, the same is true for $\{h, hg, \ldots, hg^{2n}\}$ and p_2. Let y_1 be a point on p_1 with $\mathrm{d}_S(y_1, g^n) \leq M$. By the choice of n, we have $\mathrm{d}_S(y_1, q_i) > 2\delta$ for $i = 1, 2$. Therefore by the slimness of the rectangle, there is a point y_2 be a point on p_2 with $\mathrm{d}_S(y_1, y_2) \leq 2\delta$. Now there some index $j = 0, \ldots, 2n$ such that $\mathrm{d}_S(y_2, hg^j) \leq M$.

Combining all of this, we have $\mathrm{d}_S(hg^j, g^n) \leq 2M + 2\delta$. Using that h commutes with g, this implies $\mathrm{d}_S(h, g^{n-j}) < 2M + 2\delta$. In other words, $h \in a\langle g \rangle$, where $a \in G$ is such that $|a|_S \leq 2M + 2\delta$. As S is a finite set, there are only finitely many such elements. Thus $\langle g \rangle$ has finite index in $C_G(g)$ as required. □

An immediate consequence of this is that hyperbolic groups contain no subgroups isomorphic to the Baumslag-Solitar group $\mathrm{BS}(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$, for the whole group centralises the infinite order element a^m. In particular, hyperbolic groups cannot contain any higher rank abelian subgroups, as $\mathbb{Z}^2 \cong \mathrm{BS}(1, 1)$.

Another algebraic consequence of hyperbolicity is that one has strong control over the torsion elements of the group. We examine a simple case to get an intuition for why one should be able to draw such conclusions.

Example 1.6. Let G be a group acting geometrically on a simplicial tree T (that is, a 0-hyperbolic graph), and let $H \leqslant G$ be a finite subgroup. As H is finite, the orbit Hx of any point $x \in T$ is a finite set. Thus H fixes the barycentre of Hx; it is a subgroup of a point stabiliser. Since the action is cocompact, there are finitely many conjugacy classes of point stabilisers. Moreover, since the action is proper, each point stabiliser is finite. It follows that there are only finitely many conjugacy classes of finite subgroups in G.

The general idea of the above example generalises to the hyperbolic of groups acting geometrically on hyperbolic spaces, with some complications. In trees, it is easy to define a centre for a finite set of points, while this is not so obvious in hyperbolic spaces in general.

Theorem 1.7. *Hyperbolic groups contain finitely many conjugacy classes of finite subgroups.*

Proof. Let G be a hyperbolic group with a geometric action on a δ-hyperbolic metric space X. Let $H \leqslant G$ be a finite subgroup. We will show that H preserves a *quasi-centre* of its orbits. For a bounded subset $Y \subseteq X$, denote

$$R_Y = \inf\{r > 0 \,|\, Y \subseteq B_r(x) \text{ for some } x \in X\},$$

and define the set

$$C(Y) = \{x \in X \,|\, Y \subseteq B_{R_Y+1}(x)\}.$$

This set is non-empty by definition of R_Y. We claim that $\operatorname{diam}(C(Y)) \leq 4\delta + 2$.

Let $x, x' \in C(Y)$, and let m be the midpoint of a geodesic $[x, x']$. Let $y \in Y$ be an arbitrary point in Y. By hyperbolicity, there is a point t on $[x, y]$ or $[x', y]$ with $\mathrm{d}(m, t) \leq \delta$. Suppose without loss of generality that it is the former. Now

$$\begin{aligned} \mathrm{d}(y, m) &\leq \mathrm{d}(y, t) + \mathrm{d}(t, m) \\ &\leq \mathrm{d}(y, x) - \mathrm{d}(x, t) + \delta \leq R_Y + 1 + 2\delta - \mathrm{d}(x, m). \end{aligned}$$

On the other hand, there must be some $y \in Y$ with $\mathrm{d}(y, m) \geq R_Y$. Rearranging the above equation for this y gives $\mathrm{d}(x, m) \leq 2\delta + 1$. As m is the midpoint of $[x, x']$, the claim follows.

Fix a point $x \in X$, and let $B \subseteq X$ be a compact subset such that $G \cdot B = X$, which exists as the action is cocompact. Write $K = N_{4\delta}(B)$ and note that K is also compact as X is proper. As the action is properly discontinuous, the set $T = \{g \in G \,|\, gK \cap K \neq \emptyset\}$ is finite. Thus T contains finitely many distinct subgroups.

The orbit Hx is a bounded subset of X. As the orbit Hx is setwise preserved by H, the quasi-centre $C(Hx)$ is also setwise preserved by H. Moreover, there is some $g \in G$ such that $gC(Hx) \cap B \neq \emptyset$, since $G \cdot B = X$. Thus gHg^{-1} setwise fixes the translate $gC(Hx)$. By the claim $C(Hx)$ is a set of diameter at most 4δ containing the identity, which implies that $gC(Hx) \subseteq K$. Therefore $gHg^{-1} \subseteq T$, completing the theorem. □

Hyperbolicity also allows one to rule out certain pathologies. A group in which all elements have finite order is often called a *torsion group*, or a *periodic group*. One pathology one might consider is that of being infinite while also having no elements of infinite order, that is, being an infinite torsion group. Of course, there are many silly examples of infinite torsion groups, such as an infinite direct product of finite groups, the quotient group $\mathbb{Q}/\mathbb{Z}$, or the Prüfer group $\mathbb{Z}(p^\infty)$, but sensible groups generally tend not to contain these. We will defer the proof until later on.

Theorem 1.8. *A hyperbolic group contains no infinite torsion subgroups.*

Remark 1.9. The existence of finitely generated infinite torsion groups was a for a long time a major open problem in group theory known as the *general Burnside problem*. After standing for over 60 years, a negative solution was given by Golod and Shafarevich in 1964. The groups they constructed arose in connection with the class field tower problem in number theory: they were interested in the infinitude of certain pro-p groups arising as Galois groups of certain extensions. They established a bound that relates the minimal number of relators and minimal number of generators for a finite p-group.

Another important property of hyperbolic groups is that they are, in a precise sense, very large and have many quotients. This is captured more exactly by the following theorem due to Ol'shanskii, which we is beyond the scope of this course.

Theorem 1.10. *Let G be a non-elementary hyperbolic group. For any countable group C, there is a normal subgroup $N \lhd G$ such that C is isomorphic to a subgroup of G/N.*

The property above is known as *SQ-universality*, and it satisfied by many of the generalisations of hyperbolic groups as well.