Geometric group theory
Lecture 9

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1 The space of triples

Starting with the boundary of a hyperbolic space, then we can try to reconstruct the space by considering triples of points in the boundary, viewed as vertices of ideal triangles. The subspace we get by taking 'centres' of these triangles is roughly well-defined. This is captured in the following exercise.

Exercise 1.1. Let X be a proper δ-hyperbolic metric space. Call a point $x \in X$ a *centroid* for an ideal triangle T if x is a distance of at most 10δ from each side of T. Show that every ideal triangle T has at least one centroid, and the distance between any two centroids is bounded by a constant depending only on δ. (Hint: approximate T with finite geodesic triangles, for which centroids are easy to find.)

In fact, when a group acts cocompactly on a hyperbolic space, the space of centroids is more or less the entire original space. We can view this as a sort of strong visibility property.

Exercise 1.2. Let X be a δ-hyperbolic metric space with a non-elementary,cocompact group action by isometries. Then there is some constant $K = K(\delta)$ such that every point of X is a distance of at most K from a centroid of an ideal triangle in X.

Mimicking the above, we can reconstruct a sort of abstract model for the 'interior' of an arbitrary compactum.

Definition 1.3 (Space of triples)**.** Let M be a topological space. Write $\Theta_0(M) = M^3 - \{(a,b,c) \mid \#\{a,b,c\} < 3\}$ for the space of distinct ordered triples of M, equipped with the product topology. The *space of triples* of M is the space $\Theta(M)$, obtained as the quotient of $\Theta_0(M)$ by the permutation action of the symmetric group on triples.

Remark 1.4. When M is compact and metrisable, $\Theta(M)$ is locally compact and metrisable. If a group acts on M, then there is an obvious induced action on $\Theta(M)$.

We will, for the sake of convenience, largely ignore the above formalism and refer to elements of $\Theta(M)$ as three-element subsets of M. Convergence groups and their dynamical properties can be reformulated in terms of the topology of the action on the space of triples. Of course, when M has fewer than three points, $\Theta(M)$ is empty, so we usually discard this case.

To build a dictionary between actions on M and $\Theta(M)$, we will need to translate the proper discontinuity condition back into information about sequences.

Lemma 1.5. *Let M be a compact metrisable space with at least three points, and suppose that G acts on $\Theta(M)$ properly discontinuously. Suppose $x_n \to x, y_n \to y, z_n \to z$, with $\{x, y, z\} \in \Theta(M)$. If (g_n) is a sequence of distinct group elements and $(g_n x_n), (g_n y_n)$, and $(g_n z_n)$ converge in M, at least two have a common limit point.*

Proof. Suppose otherwise, so $g_n x_n \to x', g_n y_n \to y', g_n z_n \to z'$ with $\{x', y', z'\} \in \Theta(M)$. It follows that the sequence $\{g_n x_n, g_n y_n, g_n z_n\} \to \{x', y', z'\}$ converges in $\Theta(M)$. As $\Theta(M)$ is locally compact, $\{x, y, z\}$ has a compact neighbourhood K. But then $g_m^{-1} g_n K \cap K$ is non-empty for all sufficiently large values of m and n, contradicting proper discontinuity. □

Lemma 1.6. *Let M be a compact metrisable space with at least three points, and suppose that G acts on $\Theta(M)$ properly discontinuously, and let $x_n \to x, y_n \to y, z_n \to z$ be such that $\{x, y, z\} \in \Theta(M)$. If (g_n) is a sequence in G and $a \neq b \in M$ are such that $g_n x_n \to a$, $g_n y_y \to a$, and $g_n z_n \to b$, then (g_n) has a convergence subsequence with attracting point a and repelling point z.*

Proof. Let $p \notin \{x, y, z\}$ and pass to a subsequence for which $(g_n p)$ converges. By Lemma 1.5, either $g_n p \to a$ or $g_n p \to b$. Suppose the latter is true, then pick $r \neq a, b$ and pass to a subsequence for which $g_n^{-1} r \to w$ converges. Again applying Lemma 1.5 to $(g_n x_n), (g_n z_n)$, and $(g_n^{-1} r)$ shows that $w \in \{a, b\}$. But then $r = g_n(g_n^{-1} r)$ by the same argument again converges to either a or b, a contradiction. Hence g_n converges pointwise to the constant function at a on $M - \{z\}$.

We must prove that this pointwise convergence is in fact uniform on compact subsets. Let $K \subseteq M - \{z\}$ be a compact subset and $U \subseteq M$ an open neighbourhood of a. Suppose that the convergence is not uniform, so that there is an infinite sequence (w_n) in K such that $g_n w_n \notin U$, After passing to a subsequence of (w_n), we may assume $w_n \to w \in K$. We may pass to a further subsequence for which $(g_n w_n)$ converges. Since $z \notin K$ and K is closed, $w \neq z$. Thus we can apply Lemma 1.5 to $(g_n x_n), (g_n z_n)$, and $(g_n w_n)$ to see that that $g_n w_n \to a$, a contradiction. □

Theorem 1.7. *Let M be a compact metrisable space with at least three points, G a group acting by homeomorphisms on M. Then G is a convergence group on M if and only if it acts properly discontinuously on $\Theta(M)$.*

Proof. We first prove the forward direction: suppose G is a convergence group on M. Let $K \subseteq \Theta(M)$ be a compact subset and (g_i) is an infinite sequence of elements with $g_i K \cap K \neq \emptyset$. That is, there is a sequence $(\{x_i, y_i, z_i\})$ in K such that $\{g_i x_i, g_i y_i, g_i z_i\} \in K$ for each i also. Passing to a subsequence, we may assume that $\{g_i x_i, g_i y_i, g_i z_i\}$ converges to a point $\{x', y', z'\}$ in K, by compactness.

Now, as G is a convergence group, there is a convergence subsequence (g_{n_i}) with attracting point a and repelling point b. Without loss of generality, we may suppose that x_i and y_i are not equal to b. It follows that $x' = \lim g_i x_i = a$ and $y' = \lim g_i y_i = a$, contradicting the fact that x' and y' are distinct. Thus no such infinite sequence exists.

For the converse, suppose that G acts on $\Theta(M)$ properly discontinuously. Let (g_i) be an infinite sequence of distinct elements of G. Let $\{x, y, z\} \in \Theta(M)$. By Lemma 1.5, there is some subsequence (g_{n_i}) and points a and b in M such that $g_{n_i}x \to a$, $g_{n_i}y \to a$, and $g_{n_i}z \to b$ (after possibly relabelling). If $a \neq b$, then Lemma 1.6 completes the proof, so suppose otherwise, that $a = b$. Pick some $c \neq a$ and let $w_i = g_{n_i}^{-1}c$. Passing to a further subsequence we can assume $w_i \to w$ and, after possibly relabelling, $w \neq x, y$. But then $g_{n_i}x, g_{n_i}y \to a$ but $g_{n_i}w_i = c$. Again (g_{n_i}) has a convergence subsequence by Lemma 1.6. $\square$

As a basic consequence of this characterisation, we can deduce that the only uncountable convergence groups arise in the trivial case of a group acting on a single point. One should view this as a manifestation of the essentially discrete nature of these groups. Note that this fact is certainly not obvious from the dynamical characterisation!

Corollary 1.8. *If G is a convergence group on a compact metrisable space with at least two points, then G is countable.*

Proof. Let G be a convergence group on compact metrisable space M. If M has two points, the action of G is necessarily minimal and so ΛG has two points. By an exercise from previous lectures, G has a finite index infinite cyclic subgroup, so is countable.

Suppose then that M has at least three points. Then by Proposition 1.7, G acts properly discontinuously on $\Theta(M)$, which is locally compact and metrisable. In particular, $\Theta(M)$ is locally compact, Hausdorff, and second countable. Since the action is properly discontinuous and the space is locally compact, the stabiliser of any point is finite. Moreover, the orbit of any point is discrete and second countable, and thus countable. Thus the quotient of G by the kernel of its action on $\Theta(M)$ is countable. The result now follows form the fact that the action of G on M has finite kernel. $\square$

As one might expect, more or less any dynamical criterion on a convergence group can be rephrased in terms of more topological criterion on the space of triples. For instance, being a uniform convergence group has the characterisation below, due to Tukia; the proof is somewhat long and we will not present it here.

Theorem 1.9. *Let G be a convergence group on compact metrisable space M. Then G is uniform if and only if the action of $\Theta(M)$ is cocompact.*

The latter condition is frequently given as the definition of a uniform convergence group, as the notion is based on the behaviour of cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. We will see soon that, more generally, hyperbolic groups are uniform convergence groups on the boundaries of spaces they act on geometrically.

2 Convergence groups and hyperbolic spaces

We finally turn the machinery developed in the previous lectures on groups acting on hyperbolic spaces. The key observation linking the two theories is the following.

Theorem 2.1. *Let G be a group acting properly discontinuously by isometries on a proper hyperbolic metric space X. Then the induced action of G on ∂X is a convergence action.*

Proof. Let X be a δ-hyperbolic metric space with properly discontinuous isometric G-action, and fix a basepoint $x \in X$. If G is finite there is nothing to prove, so suppose otherwise. Take (g_i) an infinite sequence of distinct elements of G. As the action is proper, the sequence $(g_i x)$ is unbounded in X. After possibly passing to a subsequence, there is some point $a \in \partial X$ such that $g_i x \to a$, since $X \cup \partial X$ is compact.

If for all sequences (b_i) in ∂X, we have $g_i b_i \to a$, then (g_i) is straightforwardly a convergence sequence with attracting and repelling point a, and so we are done. Otherwise, there is a sequence (b_i) in ∂X and $c \in \partial X$ such that $g_{n_i} b_i \to c$ with $c \neq a$, for some subsequence (g_{n_i}). By compactness of ∂X, we may pass to a further subsequence for which b_i converges to some point $b \in \partial X$. The claim is that (g_{n_i}) is a convergence sequence with attracting point a and repelling point b.

We need to show that for any compact subset $K \subseteq \partial X - \{b\}$ and any neighbourhood U of a, that $g_{n_i} K \subseteq U$ for sufficiently large i. Let K be such a subset. As ∂X is Hausdorff, K is closed and, in particular, does not intersect every neighbourhood of b. Without loss of generality, then, we may take K to be the complement of $U(b, r)$ for some $r \geq 0$. Suppose $p \notin U(b, r)$, so that $\langle p, b \rangle_x \leq r$. Note that if $q_i \to q$, then for any $\varepsilon > 0$ we have the containment $U(q_i, r + \varepsilon) \subseteq U(q, r)$ for sufficiently large i. Thus there is a sequence (y_i) in X asymptotic to p with $\langle y_i, b_i \rangle_x \leq 2r$ for all i.

Let $r \geq 0$ be large enough that $U \cap U(c, r) = \emptyset$, which exists since ∂X is normal Hausdorff. The observation above implies that $U(g_{n_i} b_i, 2r) \subseteq U(c, r)$ for sufficiently large i. Thus $\langle g_{n_i} x, g_{n_i} b_i \rangle_x \leq 2r$ for large i. It follows that, for i large enough, there is a point z_i on a geodesic $[g_{n_i} b_i, g_{n_i} x]$ with $\mathrm{d}(z_i, x) \leq 2r + 2\delta$.

Now we have

$$\langle g_{n_i} y_j, z_i \rangle_{g_{n_i} x} \leq \langle g_{n_i} y_j, g_{n_i} b_i \rangle_{g_{n_i} x} = \langle y_j, b_i \rangle_x \leq 2r,$$

where the first inequality comes from the fact that z_i is on the geodesic $[g_{n_i} b_i, g_{n_i} x]$, the equality from the fact that G acts by isometries, and the last inequality from the construction of (y_i).

Combining these with the triangle inequality implies that $\langle g_{n_i} y_j, x \rangle_{g_{n_i} x} \leq 6r + 4\delta$. As $\mathrm{d}(g_{n_i} y_j, x)$ is unbounded while $\langle g_{n_i} y_j, x \rangle_{g_{n_i} x}$ is bounded, we must have that $\langle g_{n_i} y_j, g_{n_i} x \rangle_x$ is unbounded in i. That is, for any $r' \geq 0$, there is i such that $\langle g_{n_i} y_j, g_{n_i} x \rangle_x \geq r'$. Being that $(g_{n_i} y_j)_{j \in \mathbb{N}}$ is asymptotic to $g_{n_i} p$, this means $g_{n_i} p \in U(a, r')$. Choosing $r' \geq 0$ large enough so that $U(a, r') \subseteq U$ thus completes the proof. $\square$